Aryabhatta Knowledge University (AKU)

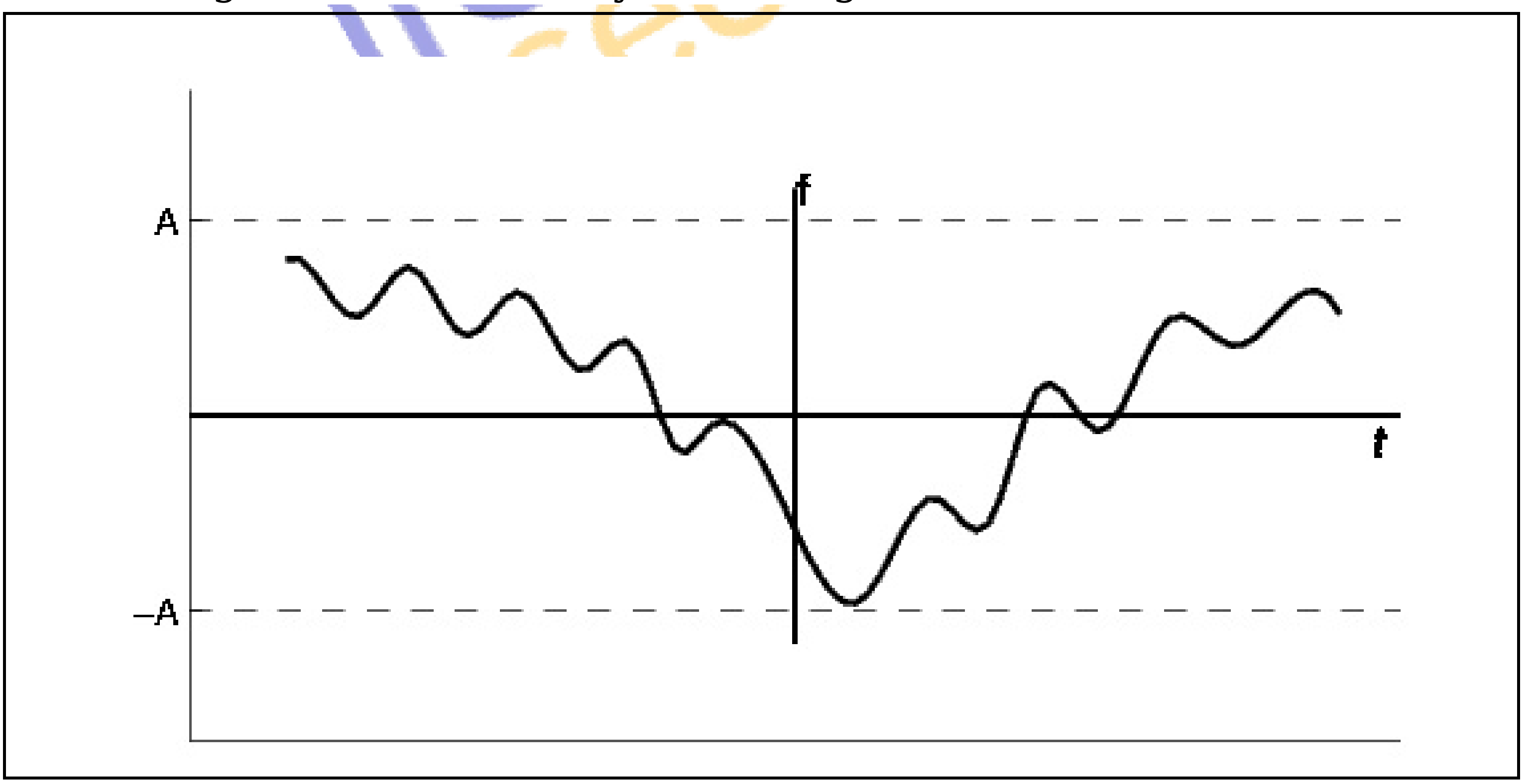
ECE

Signal and System

Solved Exam Paper 2019

Question. Derive the condition of BIBO stability. What is an invertible system?

Answer: A bounded signal is any signal such that there exists a value such that the absolute value of the signal is never greater than some value. Since this value is arbitrary, what we mean is that at no point can the signal tend to infinity, including the end behavior.



Now that we have identified what it means for a signal to be bounded, we must turn our attention to the condition a system must possess in order to guarantee that if any bounded signal is passed through the system, a bounded signal will arise on the output. It turns out that a continuous time LTI system with impulse response h(t) is BIBO stable if and only if

$$\int_{-\infty}^{\infty} |h(t)| \, \mathrm{d}t < \infty$$

Continuous-Time Condition for BIBO Stability

Stability is very easy to infer from the pole-zero plot of a transfer function. The only condition necessary to demonstrate stability is to show that the

iω

ω-axis is in the region of convergence. Consequently, for stable causal systems, all poles must be to the left of the imaginary axis.

Bounded input, bounded output (BIBO) stability is a form of stability often used for

signal processing applications. The requirement for a linear, shift invariant, discrete

time system to be BIBO stable is for the output to be bounded for every input to

the system that is bounded. Assume that R is a real or complex number with finite

magnitude such that

$|R| < \infty$.

A discrete time system will be BIBO stable if for all inputs, x(n), such that

$|x(n)| < |R| < \infty$,

the output, y(n), satisfies

$|y(n)| < |R| < \infty$.

Based on property of memory, system can be classified as Static and Dynamic Systems. If the System for its functionality requires memory for its operation, it is known as dynamic system otherwise it is known as static system. Invertible and Non-Invertible System: To understand the concept of Invertibility, let us consider two systems: System-1 and System-2. Let x(t) and y(t) be the input and the output of system-1 respectively. The output of system-1 , y(t), is given as input to system-2 whose output is w(t). System-1 can be said to be invertible if output of system-2 ,w(t) is same as x(t). If there is no possibility of w(t) being x(t), then system-1 is a non-invertible Systems. It is observed that when distinct inputs lead to distinct outputs, such a system is automatically a invertible system

Question. Compute the trigonometric Fourier coefficients of a periodic signal?

Answer: Fourier series in trigonometric form can be easily derived from its exponential form.

The complex exponential Fourier series representation of a periodic signal x(t) with fundamental period To is given by

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_o t} \quad \omega_o = \frac{2\pi}{T_o}$$

Since sine and cosine can be expressed in exponential form. Thus by manipulating the exponential Fourier series, we can obtain its

Trigonometric form.

The **trigonometric Fourier series** representation of a periodic signal x (t) with fundamental period T, is given by

$$x(t) = a_o + \sum_{k=1}^{\infty}(a_k cosk\omega_o t + b_k sink\omega_o t) \quad \omega_o = \frac{2\pi}{T_o}$$

Where ak and bk are Fourier coefficients given by

$$a_k = \frac{2}{T_o}\int_{T_o} x(t)cosk\omega_o tdt, \quad b_k = \frac{2}{T_o}\int_{T_o} x(t)sink\omega_o tdt$$

a0 is the dc component of the signal and is given by

$$a_o = \frac{1}{T_o}\int_{T_o} x(t)dt$$

Relationship between coefficients of exponential form and coefficients of trigonometric form

$$a_0 = C_0,\ a_k = C_k + C_{-k},\ b_k = j(C_k - C_{-k}),$$
$$c_k = \frac{1}{2}(a_k - jb_k),\ c_{-k} = \frac{1}{2}(a_k + jb_k)$$

When x (t) is real, then a, and b, are real, we have

$$a_k = 2Re[c_k] \ \ and \ \ b_k = -2Im[c_k]$$

Question. State and prove Perceval's theorem

Answer: Perceval's theorem expresses the energy of a signal in time-domain in terms of the average energy in its frequency components.

Suppose if the x[n] is a sequence of complex numbers of length N : *xn={x0,x1,…,xN-1}*, its N-point discrete Fourier transform (DFT): *Xk={X0,X1,…,XN-1}* is given by

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

The inverse discrete Fourier transform is given by

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}$$

Suppose if *x[n]* and *y[n]* are two such sequences that follows the above definitions, the Parseval's theorem is written as

$$\boxed{\sum_{n=0}^{N-1} x[n] y^*[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] Y^*[k]}$$

where, $*$ indicates conjugate operation.

Deriving Parseval's theorem

$$\sum_{n=0}^{N-1} x[n]y^*[n] = \sum_{n=0}^{N-1} x[n] \left(\frac{1}{N} \sum_{k=0}^{N-1} Y[k] e^{j\frac{2\pi}{N}kn} \right)^*$$

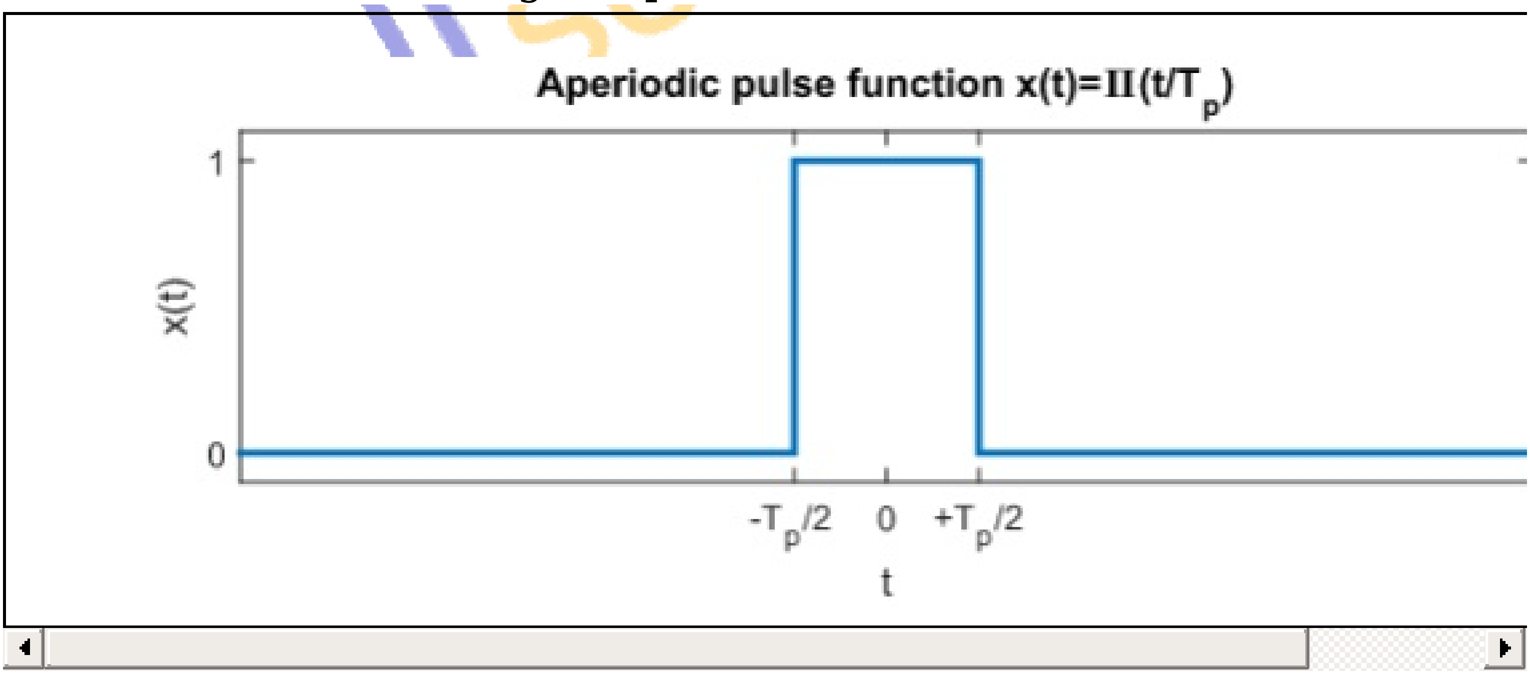
$$= \frac{1}{N} \sum_{n=0}^{N-1} x[n] \sum_{k=0}^{N-1} Y^*[k] e^{-j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} Y^*[k] \cdot \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] Y^*[k]$$

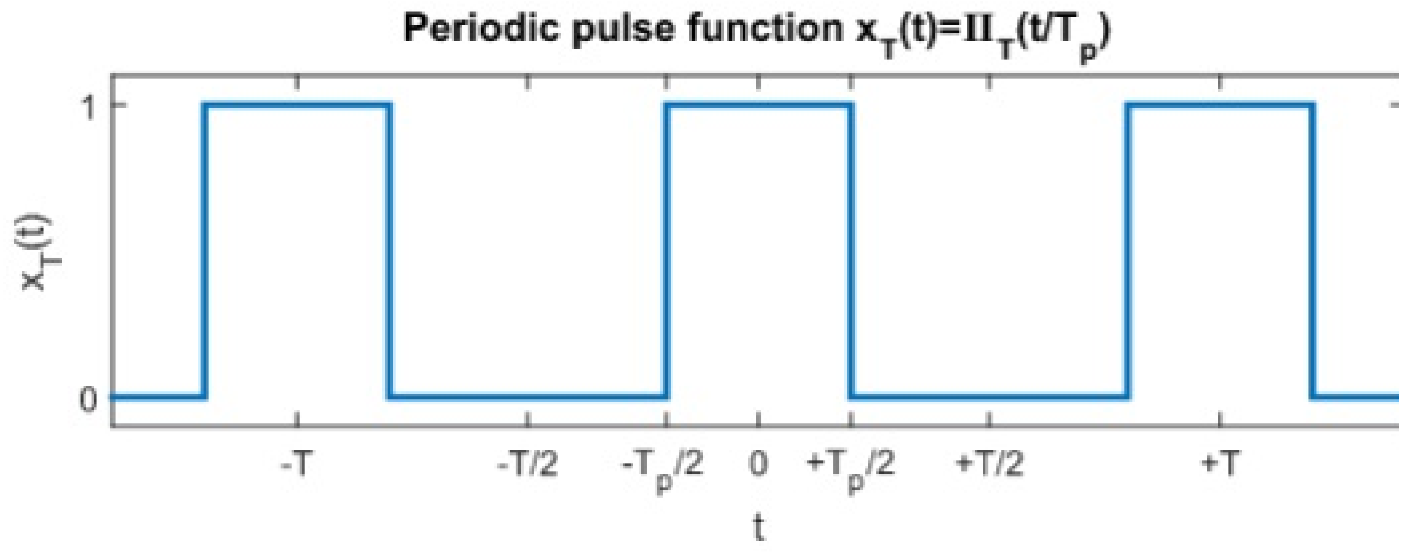
Question. Establish the relation between Fourier series and Fourier Transform

Answer: Consider an aperiodic function, *x(t)*, of finite extent (i.e., it is only non-zero for a finite interval of time). In the diagram below this function is a rectangular pulse.



The periodic extension of *x(t)* is called $x_T(t)$, and is just *x(t)* replicated every *T* seconds, such that it is periodic with period *T* (i.e.,

$x_T(t+nT)=x(t)$, with n an integer).



The question to be answered is: how are the Fourier Transform of $x(t)$ and the Fourier Series for $x_T(t)$ related?

The Fourier Transform of $x(t)$ is $X(\omega)$ and is given by:

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

The Fourier Series coeffients, c_n, of $x_T(T)$ are given by:

$$c_n = \frac{1}{T}\int_T x_T(t)e^{-jn\omega_0 t}dt = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} x_T(t)e^{-jn\omega_0 t}dt$$

If the function $x(t)$ is only non-zero in the inerval $(-T/2, +T/2)$ we can the Fourier Transform as

$$X(\omega) = \int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t) e^{-j\omega t} dt$$

Comparing the equations and noting the $x_T(t) = x(t)$ over the inve integration, we can see that the relationship between the Fourie coefficients, c_n, and the Fourier Transform, $X(\omega)$, is given by

$$c_n = \frac{1}{T} X(n\omega_0)$$

In the particular example of the rectangular pulse

$$X(\omega) = T_p \text{sinc}\left(\frac{\omega T_p}{2\pi}\right)$$

We could also get c_n directly from $X(\omega)$:

$$\frac{1}{T} X(n\omega_0) = \frac{1}{T}\left(T_p \text{sinc}\left(\frac{n\omega_0 T_p}{2\pi}\right)\right) = \frac{T_p}{T}\text{sinc}\left(\frac{n\frac{2\pi}{T}T_p}{2\pi}\right) = \frac{T_p}{T}\text{sinc}\left(\frac{nT_p}{T}\right)$$

Question. Establish the relation between Laplace transform and z- transform

Answer: The Laplace transform converts differential equations into algebraic equations. Whereas the Z-transform converts difference equations (discrete versions of differential equations) into algebraic equations.

The Laplace transform maps a continuous-time function f(t) to f(s) which is defined in the s-plane. In the s-plane, s is a complex variable defined as:

$$s = \sigma + j\Omega$$

Similarly, the Z-transform maps a discrete time function f(n) to f(z) that is defined in the z-plane. Here z is a complex variable defined as:

$$z = re^{j\omega}$$

Consider a periodic train of impulses p(t) with a period T.

$$P(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Now consider a periodic continuous time signal x(nT).

Take a product of the above two signals as shown below.

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT).\delta(t - nT)$$

Taking Laplace transform of the above signal and using the identity

$$L(\delta(t - nT)) = e^{-nsT}$$

Thus,

$$x_p(s) \quad = \quad \sum_{n=-\infty}^{\infty} x(nT).e^{-nsT}$$

Which can be written as:

$$x_p(s) \quad = \quad \sum_{n=-\infty}^{\infty} x(nT).(e^{st})^{-n}$$

Compare this equation with that of z-transform

$$X(Z) \quad = \quad \sum_{n=-\infty}^{\infty} x(n).z^{-n}$$

Thus we finally get the relation:

$$z \quad = \quad e^{sT}$$

Another representation:

$$s \quad = \quad \frac{2(z-1)}{T(z+1)}$$

Question. List down the properties of ROC

Answer: Properties of ROC of Laplace Transform:

1. The ROC of X(s) consists of strips parallel to the jω axis in the s-plane.

2. The ROC does not contain any poles.

3. If x(t) is of finite duration and is absolutely integrable, then the ROC is the entire s-plane.

4. It x(t) is a right sided signal, that is x(t) = 0 for $t<t0<\infty$ then the

ROC is of the form Re(s)> amax , where a max equals the maximum real part of any of the poles of X(s).

5. If x(t) is a left sided, that is x(t) = 0 for t>t1> -∞, then the ROC is of the form Re(s)< amin , where amin equals the minimum real part of any of the poles of X(s).

6. If x(t) is a two sided signal, than the ROC is of the form a1<Re(s)< a2